

Tangent Lie algebras of derived stacks.

Idea (Kapranov 99)

$X$  Kähler manifold:  $H^*(T_X)$  has a Lie structure up to homotopy  $(L_\infty)$ .

The 2-bracket is the Atiyah class of  $T_X$ :

$$T_X \rightarrow T_X \otimes \Omega_X[1].$$

Question: How to prove such a statement in (derived) alg. geom.  $\mathcal{O}_X$ -linear?

→ write down the Atiyah class and the upper brackets + check compatibilities ...  $\ddot{\smile}$

rather

→ Find a conceptual approach:

Deformation Theory.

(Hinich-Lurie-Bridgman...).

(check  $k=0$ )

$$\left\{ \begin{array}{l} \text{Lie algebras up} \\ \text{to homotopy} \end{array} \right\} / k \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{Formal moduli} \\ \text{problems} \end{array} \right\} / k$$

$$\left\{ \begin{array}{l} \text{Lie algebras up} \\ \text{to homotopy} \end{array} \right\} / \mathcal{O}_X \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{Formal moduli} \\ \text{problems} \end{array} \right\} / X$$

↖ choose the right one



From a base field to a base algebra:

- CDGA: Building blocks of derived alg. geom.:

$$\dots \rightarrow A' \rightarrow A^0 \rightarrow 0, \quad \mu: A \otimes A \rightarrow A \quad \begin{array}{l} \text{symmetric} \\ \text{associative} \end{array} \quad (\Delta \text{ sign rule})$$

up to qiso.

$\rightarrow$  Model category  $\rightarrow$   $(\infty, 1)$ -category.

- $\text{FTP}_A$ : "geometric objects encoding the local information around an  $A$ -point".

Ex  $\text{Spec}(A[t]/t^2), \text{Spec}(A[t]/t^n)$

Functor of points: • Geometric obj. over  $A$  with an  $A$ -point

$$\text{cdga}_{A/A} \rightarrow \text{Sets}_S + \text{properties}$$

- Only infinitesimal information:  
restrict to a full sub-category of  $\text{cdga}_{A/A}$ .

$\rightarrow$  two approaches: • what is this infinitesimal information?  
• Restrict to Artinian  $\text{cdga}$  ("small")  
• what happens on the Lie-side?

2)



$\bullet \text{dglie}_A = \text{Lie}(\text{dgMod}_A)$

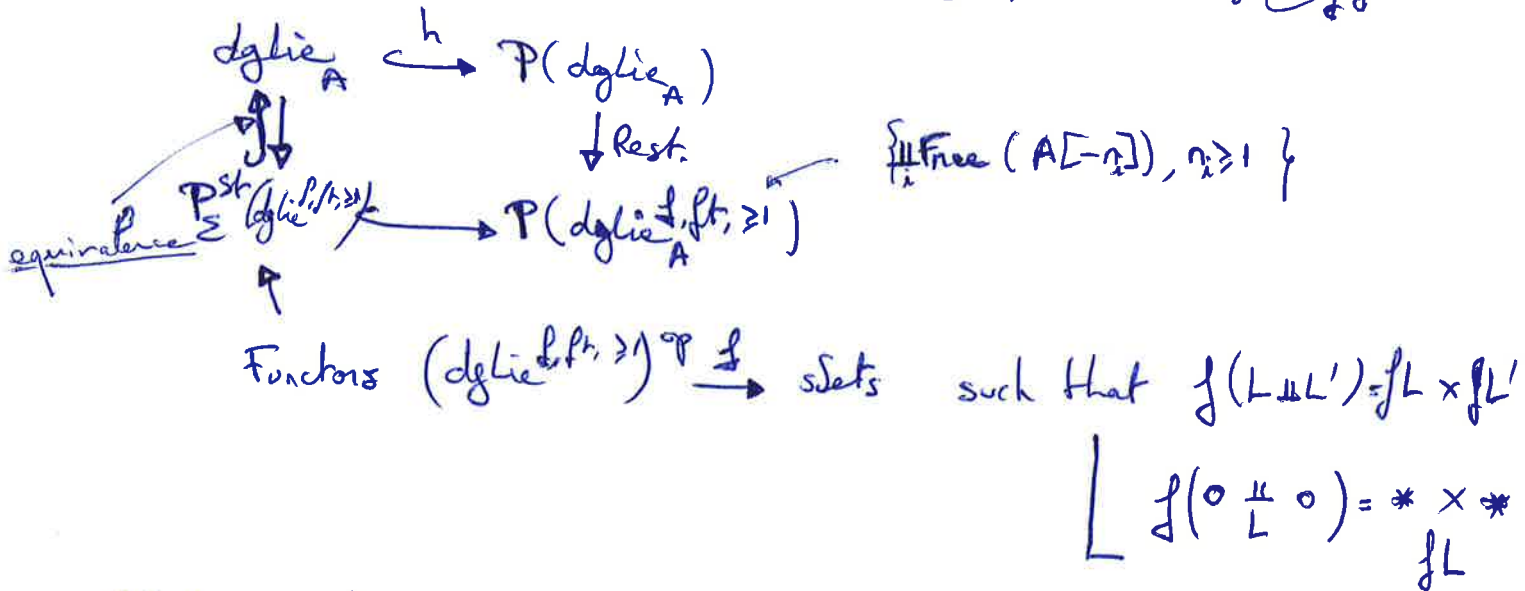
$$\text{dgMod}_A \left\{ \begin{array}{l} \cdots \rightarrow L^{-1} \rightarrow L^0 \rightarrow L^1 \rightarrow \cdots \\ A \otimes L \rightarrow L \quad \text{actions} \\ L \otimes L \rightarrow L \quad \text{bracket} \end{array} \right.$$

- $\bullet$  A-linear
- $\bullet$  skew-symmetric
- $\bullet$  Jacobi

$\Delta$  Sign Rule

Up to qisa  $\rightarrow$  model  $\setminus (\infty, 1)$ -category.

Take it closer to  $\text{Frip}_A$ : write  $\text{dglie}_A$  as a category of functors.

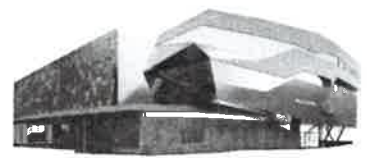


Algebraic Theory:  $\rightarrow \text{cdga}_K^{\leq 0} = \mathcal{P}_\Sigma(\text{Poly})$

$\rightarrow \text{dgMod}_A = \mathcal{P}_\Sigma^{\text{st}}(\text{dgMod}_A^{f, ft, \geq 1}) \rightarrow f(\pi[\pm 1]) = \text{sef}(M)$

$\rightarrow$  Want to write  $\text{Frip}_A = \mathcal{P}_\Sigma^{\text{st}}(???)$

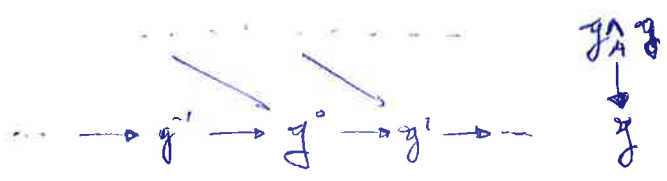
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• Link Lie  $\leftrightarrow$  FHP: Chevalley-Eilenberg cohomology

$$\left( (\text{Sym } \mathfrak{g}[1])^*, d \right) \rightarrow A \otimes \mathfrak{g} \xrightarrow{\text{qprg}} \mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{k}^*$$

Same idea over  $A$ , with a dg Lie



$$C_A: \text{dg Lie}_A \rightarrow (\text{cdga}_{A/A})^{\text{op}}$$

[def. with model cats + preserves qiso. as in the case of a field.]

Lemma: The functor  $C_A$  maps colimits to limits.  
 $\implies$  Right adjoint  $D_A$   
 $\implies$  Enough to know what happens with  $\text{dg Lie}_{\mathbb{Z}, \mathbb{Z}, \mathbb{Z}}$ .

$$\text{Free} \left( \underbrace{\bigoplus_{i=1}^r A^{n_i}[-i]}_{\mathfrak{n}} \right) \quad \underline{n_i \text{ almost 0.}}$$

Lemma:  
 $C_A(\text{Free}(\mathfrak{n})) = A \oplus \mathfrak{n}^r[-1] \leftarrow \overline{\text{univ. sq. o extension.}}$   
Rmk:  $\mathfrak{n}^r[-1]$  in degrees  $[-\infty, 0]$

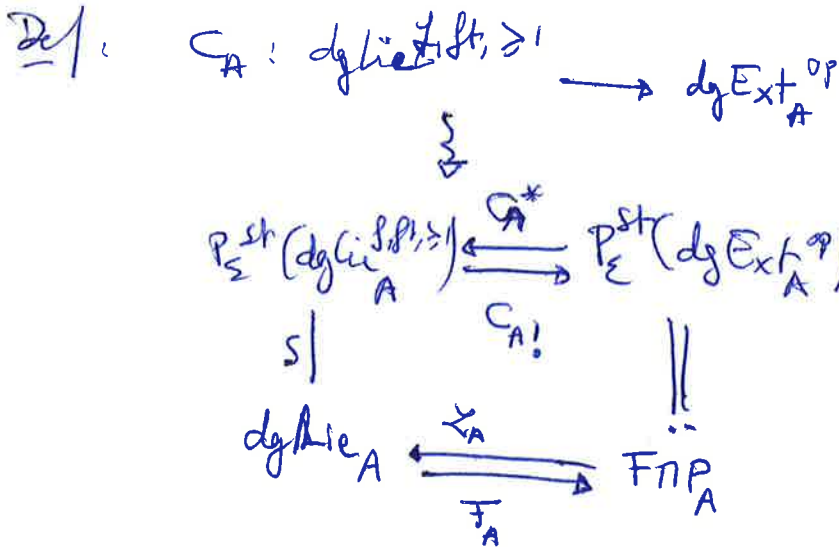
Def:  $\text{dg Ext}_A^{\leq 0} C$   $\text{cdga}_{A/A}^{\leq 0}$  spanned by  $A \oplus \mathfrak{N}$  where  $\mathfrak{N}$  free  $A$ -dg module, of  $\mathfrak{k}$ , in neg. degrees.

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Def:  $FNP_A = P_{\Sigma}^{st}(dgExt_A^{op}) = \left\{ \begin{array}{l} dgExt_A \xrightarrow{\text{sets}} \\ f(A \otimes \Omega) = f(A \otimes \Omega) \times f(A \otimes \Omega) \\ f(A \otimes \Omega) = \Omega f(A \otimes \Omega) \end{array} \right.$

Remark: difference with Lurie's approach: small \(\mathbb{A}\)-algebra's  
 • add some test elements  $dgExt \rightarrow dgArt$   
 • add some conditions (Schlesinger) ) the cats are equivalent.



⚠ In general, not an equivalence

Remark:  $L = \text{Free}(A^2[-2])$

The adjunction map  $L \rightarrow \sum_{A} F_A L$  is  $L \rightarrow (L^v)^v$  ( $H^0 A$  noetherian)

$$L = \bigoplus_{i \geq 0} A^{\oplus i}[i]$$

finite amount in every degree...

$$L^v = \prod_{i \geq 0} A^{\oplus i}[i] = \bigoplus_{i \geq 0} A^{\oplus i}[i]$$

but  $(L^v)^v = \prod_{i \geq 0} A^{\oplus i}[i] \neq L$

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III gluing

Lemma:  $A \rightarrow B$  map of edga's.

$$\begin{array}{ccc}
 \text{dglie}_A & \xrightarrow{F_A} & \text{FNP}_A \\
 \downarrow \otimes_B^A & \curvearrowright & \downarrow \text{induced by edga } A/A \xrightarrow{\otimes_B^A} \text{edga } B/B \\
 \text{dglie}_B & \xrightarrow{F_B} & \text{FNP}_B
 \end{array}$$

Def:  $X$  derived stack "geometric" (namely Artin loc. of fp.)

- $\text{dglie}_X = \varinjlim_{\text{Spec } A \rightarrow X} \text{dglie}_A$  coh. sheaves of  $\mathcal{O}_X$ -linear dglie algebras.
- $\text{FNP}_X = \varinjlim_{\text{Spec } A \rightarrow X} \text{FNP}_A$  sheaves of FNP.

Def:  $X \rightarrow Y \xrightarrow{\text{Artin } \dots} X$ ,  $\text{Spec } A \xrightarrow{u} X$

Let  $\hat{Y}_X/A$  be the FNP/A:

$$\begin{array}{ccc}
 \text{dglie}_{\text{Ext}_A} & \rightarrow & \text{sets} \\
 A \rightarrow B \rightarrow A & \leftrightarrow & \text{Rap}_{\text{Spec } A / - / \text{Spec } A} (\text{Spec } B, Y_X \text{Spec } A) \\
 & & \parallel \\
 & & \text{Rap}(\Pi^Y, T_{Y/X, u})
 \end{array}$$

if  $B = A \oplus \mathcal{O}_U$

3  $Y \rightarrow \hat{Y}_X$

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Def: Let  $X$  be a derived Artin stack /k char k = 0

$$\begin{array}{c} X \\ \downarrow A \\ X \times X \\ \downarrow L_P \\ X \end{array}$$

$\longrightarrow$

$\widehat{X \times X}$

$\longrightarrow$

$\mathcal{P}_X$

$\longrightarrow$

$\mathbb{T}_X[-1]$

(actually

$$\left[ \mathbb{T}_{X \times X/X} \xrightarrow{\sim} \mathbb{T}_X \right]$$

$\mathrm{FPP}_X$

$\longrightarrow$

$d\mathrm{y}\mathrm{l}\mathrm{i}\mathrm{e}_X$

$\longrightarrow$

$d\mathrm{g}\mathrm{D}\mathrm{o}\mathrm{d}_X$

$\cong \mathrm{Q}\mathrm{c}\mathrm{o}\mathrm{h}(X)$

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