Tangent Lie algebras of derived stacks.

Idea (Kapranov 99)

X Kähler manifold: $H^{\bullet-1}(T_X)$ has a Lie structure up to homot
($\mathcal{L}_\infty$-).

The 2-bracket is the Atiyah class of T_X:

$$T_X \longrightarrow T_X \otimes \Omega_X[1].$$

Question: How to prove such a statement in (derived) alg. geom
$\mathcal{O}_X$-linear?

→ write down the Atiyah class and the upper brackets
+ check compatibilities ... :(

rather → Find a conceptual approach:
Deformation theory.

(Hinich - Lurie - Pridham ...).

(char $k = 0$)

$$\left\{\begin{array}{l}\text{Lie algebras up} \\ \text{to homotopy}\end{array}\right\}_{/k} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{Formal moduli} \\ \text{problems}\end{array}\right\}_{/k}.$$

$$\left\{\begin{array}{l}\text{Lie algebras up} \\ \text{to homotopy } /\mathcal{O}_X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Formal moduli} \\ \text{problems } /X\end{array}\right\}$$

↑ choose the right one

II From a base field to a base cdga:

- CDGA: Building blocks of derived alg. geom.:

  $\cdots \to A^{-1} \to A^0 \to 0$, $\mu : A \otimes A \to A$ symmetric (⚠ sign rule) associative.

  up to qiso.

  $\to$ Model category $\rightsquigarrow$ $(\infty,1)$-category.

- FMP_A: "Geometric objects encoding the local information around an A-point".

Ex $\operatorname{Spec}(A[t]/t^2)$, $\operatorname{Spec}(A[t]/t^n)$ $\rightsquigarrow$

Functor of points: • Geometric obj. over A with an A-point.

$\mathrm{cdga}_{A/A} \to \mathrm{sSets}$ + properties
S

• Only infinitesimal information:
restrict to a full sub-category of $\mathrm{cdga}_{A/A}$.

$\rightsquigarrow$ two approaches: • what is this infinitesimal information?
$\rightsquigarrow$ Restrict to Artinian cdga ("small")

• what happens on the Lie-side?

31 Caroline St. N., Waterloo
Ontario, Canada, N2L 2Y5

Phone (519) 569-7600
Fax (519) 569-7611

www.perimeterinstitute.ca

- $\mathrm{dgLie}_A = \mathrm{Lie}(\mathrm{dgMod}_A)$

$$\mathrm{dgMod}_A \left\{ \cdots \longrightarrow L^{-1} \longrightarrow L^{0} \longrightarrow L^{1} \longrightarrow \cdots \right.$$

$A \otimes L \longrightarrow L$ action

$L \otimes L \longrightarrow L$ bracket: · A-linear
· skew-symmetric
· Jacobi
} ⚠ Sign Rul

Up to qiso ⇝ model \ $(\infty,1)$-category.

Make it closer to FMP_A: write dgLie_A as a category of functors.

$$\begin{tikzcd}
\mathrm{dgLie}_A \arrow[r, hook, "h"] \arrow[d, shift left] & \mathcal{P}(\mathrm{dgLie}_A) \arrow[d, "\mathrm{Rest.}"] \\
\mathcal{P}^{st}_{\Sigma}(\mathrm{dgLie}_A^{f,ft,\geq 1}) \arrow[u, shift left] \arrow[r, hook] & \mathcal{P}(\mathrm{dgLie}_A^{f,ft,\geq 1})
\end{tikzcd}$$

equivalence

$\{\coprod_i \mathrm{Free}(A[-n_i]),\ n_i \geq 1\}$

Functors $(\mathrm{dgLie}^{f,ft,\geq 1})^{op} \xrightarrow{f} \mathrm{sSets}$ such that $f(L \amalg L') = fL \times fL'$

$f(0 \amalg_L 0) = * \times_{fL} *$

Algebraic Theory: $\longrightarrow \mathrm{cdga}_k^{\leq 0} = \mathcal{P}_{\Sigma}(\mathrm{Poly})$

$\longrightarrow \mathrm{dgMod}_A = \mathcal{P}^{st}_{\Sigma}(\mathrm{dgMod}_A^{f,ft,\geq 1})$ ⇝ $f(M[-1]) = \Omega f(M)$

⇝ Want to write $\mathrm{FMP}_A = \mathcal{P}^{st}_{\Sigma}(???)$.

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• Link Lie ↔ FMP: Chevalley-Eilenberg cohomology

$$\left((\mathrm{Sym}\, \mathfrak{g}[1])^*, d\right) = \left(\cdots \to \wedge^3 \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g} \xrightarrow{0} k\right)^*$$

Same idea over A, with a dg Lie

$$\mathfrak{g}\wedge_A \mathfrak{g} \to \mathfrak{g}$$

$$\cdots \to \mathfrak{g}^{-1} \to \mathfrak{g}^{0} \to \mathfrak{g}^{1} \to \cdots$$

$$C_A : \mathrm{dgLie}_A \longrightarrow (\mathrm{cdga}_{A/A})^{op}$$

[def. with model cats + preserves qiso. as in the case for field.]

Lemma: The functor C_A maps colimits to limits.

⇝ Right adjoint D_A

⇝ Enough to know what happens with $\mathrm{dgLie}^{f, ft, \geq 1}$.

$$\mathrm{Free}\Big(\underbrace{\bigoplus_{i=1}^{\#} A^{n_i}[-i]}_{n}\Big) \qquad n_i \text{ almost } 0.$$

Lemma:

$C_A(\mathrm{Free}(n)) = A \oplus n^\vee[-1]$ ← triv. sq. 0 extension.

Rmk: $n^\vee[-1]$ in degrees $]-\infty, 0]$

Def: $\mathrm{dgExt}_A \subset \mathrm{cdga}^{\leq 0}_{A/A}$ spanned by $A \oplus N$ where N free A-dg module, of ft, in neg. degrees.

Def: $FMP_A = \mathcal{P}_\Sigma^{st}(dgExt_A^{op}) = \left\{ dgExt_A \xrightarrow{f} sSets \;\middle|\; \begin{array}{l} f(A\oplus M \oplus N) = f(A\oplus M) \times f(A\oplus \ldots \\ f(A\oplus M[1]) = \Omega f(A\oplus M \ldots \end{array} \right.$

Rmk: difference with Lurie's approach: small \ Artinian cdga's
- add some test elements $dgExt \rightsquigarrow dgArt$.
- add some conditions (Schlessinger)

) the cats are equivalent.

Def: $C_A : dgLie_A^{f.d., \geq 1} \longrightarrow dgExt_A^{op}$

$$\begin{tikzcd}
\mathcal{P}_\Sigma^{st}(dgLie_A^{f.d.,\geq 1}) \arrow[r, shift right, "C_{A!}"'] \arrow[d, "S"', "\wr"] & \mathcal{P}_\Sigma^{st}(dgExt_A^{op}) \arrow[l, shift right, "C_A^*"'] \arrow[d, equal] \\
dgLie_A \arrow[r, shift right, "F_A"'] & FMP_A \arrow[l, shift right, "\mathcal{L}_A"']
\end{tikzcd}$$

⚠ In general, not an equivalence

Rmk: $L = \mathrm{Free}(A^2[-1])$

The adjunction map $L \longrightarrow \mathcal{L}_A F_A L$ is $L \longrightarrow (L^\vee)^\vee$ ($H^0 A$ noethe…

$L = \bigoplus_{i>0} A^{n_i}[i]$

finite amount in every degree…

$L^\vee = \prod_{i>0} A^{n_i}[i] = \bigoplus_{i>0} A^{n_i}[i]$ but $(L^\vee)^\vee = \prod_{i>0} A^{n_i}[-i] \neq L$.

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II) Gluing

<u>Lemma</u>: $A \to B$ map of cdga's.

$$\begin{tikzcd} \mathrm{dgLie}_A \arrow[r, "\mathcal{F}_A"] \arrow[d, "- \otimes_A B"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{FMP}_A \arrow[d] \\ \mathrm{dgLie}_B \arrow[r, "\mathcal{F}_B"'] & \mathrm{FMP}_B \end{tikzcd}$$

$\leftarrow$ induced by $\mathrm{cdga}_{A/A} \xrightarrow{- \otimes_A B} \mathrm{cdga}_{B/B}$

Def: X derived stack "geometric" (namely Artin loc. of f.p.)

- $\mathrm{dgLie}_X = \lim_{\mathrm{Spec} A \to X} \mathrm{dgLie}_A$ qcoh. sheaves of $\mathcal{O}_X$-linear dgLie algebras.

$\mathcal{F}_X \downarrow \uparrow \mathcal{L}_X$

- $\mathrm{FMP}_X = \lim_{\mathrm{Spec} A \to X} \mathrm{FMP}_A$ sheaves of FMP.

Def: $X \to Y \xrightarrow{\underset{\text{Artin} \ldots}{f}} X$, $\mathrm{Spec}\, A \xrightarrow{u} X$

Let $\hat{Y}_X/_A$ be the FMP $/A$: $\mathrm{dgExt}_A \to \mathrm{sSet}$.

$A \to B \to A \longmapsto \mathrm{Map}_{\mathrm{Spec} A/-/\mathrm{Spec} A}(\mathrm{Spec}\, B, Y \times_X \mathrm{Spec}\, A)$

$\| $

$\mathrm{Map}(M^\vee, T_{Y/X,u})$

if $B = A \oplus M$

$Y \longmapsto \hat{Y}_X$

3

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Def: Let X be a derived Artin stack $/k$ char $k = 0$

$$\begin{array}{c} X \\ \downarrow \Delta \\ X\times X \\ \downarrow p_1 \\ X \end{array} \longmapsto \widehat{X\times X} \longmapsto \ell_X \longmapsto T_X[-1]$$

$\left[\begin{array}{l}\text{actually} \\ T_{X\times X/X} \simeq T_X\end{array}\right.$

$$\mathrm{FMP}_X \longrightarrow \mathrm{dgLie}_X \longrightarrow \mathrm{dgMod}_X$$
$$\shortparallel$$
$$\mathrm{Qcoh}(X)$$

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